

Exponential mixing for stochastic model of two-dimensional second grade fluids

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Abstract: In this paper, we establish the exponential mixing property of stochastic models for the incompressible second grade fluid. The general criterion established by Cyril Odasso [15] plays an important role.

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1 Introduction

In this paper, we are concerned with the exponential mixing property of stochastic models for the incompressible second grade fluid which is a particular class of Non-Newtonian fluid. Let \mathcal{O} be a connected, bounded open subset of \mathbb{R}^2 with boundary $\partial\mathcal{O}$ of class \mathcal{C}^3 . We consider equation

$$\begin{aligned} d(u - \alpha\Delta u) + \left(-\nu\Delta u + \operatorname{curl}(u - \alpha\Delta u) \times u + \nabla\mathfrak{P} \right) dt \\ = \phi(u)dW, \quad \text{in } \mathcal{O} \times (0, \infty), \end{aligned} \quad (1.1)$$

under the following condition

$$\begin{cases} \operatorname{div} u = 0 \text{ in } \mathcal{O} \times (0, \infty); \\ u = 0 \text{ in } \partial\mathcal{O} \times [0, \infty); \\ u(0) = u_0 \text{ in } \mathcal{O}, \end{cases} \quad (1.2)$$

where $u = (u_1, u_2)$ and \mathfrak{P} represent the random velocity and modified pressure, respectively. W is a cylindrical Wiener process on a Hilbert space U defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, and $\phi(u)dW$ represents the external random force.

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The interest in the investigation of the second grade fluids arises from the fact that it is an admissible model of slow flow fluids, which contains a large class Non-Newtonian fluids such as industrial fluids, slurries, polymer melts, etc.. Furthermore, the second grade fluid has general and pleasant properties such as boundedness, stability, and exponential decay (see [7]). It also has interesting connections with many other fluid models, see [13, 2, 3, 19, 20, 11, 12] and references therein. For example, it can be taken as a generalization of the Navier-Stokes Equation. Indeed equation (1.1) reduces to Navier-Stokes equation when $\alpha = 0$. Furthermore, it was shown in [13] that the second grade fluids models are good approximations of the Navier-Stokes equation. Finally we refer to [14, 7, 8, 9] for a comprehensive theory of the second grade fluids.

The stochastic model of two-dimensional second grade fluids (1.1) has been recently studied in [18], [16] and [17], where the authors obtained the existence and uniqueness of solutions and investigated the behavior of the solution as $\alpha \rightarrow 0$. We mention that the martingale solution of the system (1.1) driven by Lévy noise are studied in [10].

In this paper, we establish the exponential mixing property of stochastic models for the incompressible second grade fluid driven by multiplicative, but possibly degenerate noise. The exponential mixing characterizes the long time behaviour of the solutions of the stochastic partial differential equations. More precisely, under reasonable conditions, we showed the equation (1.1) has a unique invariant measure, and the law of the solution converges to the invariant measure exponentially fast. We will apply the criterion established in [15] by Cyril Odasso. To this end, we need to prove the exponential integrability of certain energy functionals of the solutions, which is non-trivial.

This article is divided into four sections. In Section 2, we present some preliminaries. Section 3 is devoted to the formulation of the main result. The proof of our main result is given in Section 4.

2 Preliminaries

In this section, we will introduce some functional spaces and preliminary facts which will be used later.

Let $1 \leq p < \infty$, and let k be a nonnegative integer. We denote by $L^p(\mathcal{O})$ and $W^{k,p}(\mathcal{O})$ the usual L^p and Sobolev spaces, and write $W^{k,2}(\mathcal{O}) = H^k(\mathcal{O})$. Let $W_0^{k,p}(\mathcal{O})$ be the closure in $W^{k,p}(\mathcal{O})$ of $\mathcal{C}_c^\infty(\mathcal{O})$ the space of infinitely differentiable functions with compact support in \mathcal{O} . We denote $W_0^{k,2}(\mathcal{O})$ by $H_0^k(\mathcal{O})$. We endow the Hilbert space $H_0^1(\mathcal{O})$ with the scalar product

$$((u, v)) = \int_{\mathcal{O}} \nabla u \cdot \nabla v dx = \sum_{i=1}^2 \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \quad (2.3)$$

where ∇ is the gradient operator. The norm $\|\cdot\|$ generated by this scalar product is equivalent to the usual norm of $W^{1,2}(\mathcal{O})$ in $H_0^1(\mathcal{O})$.

In what follows, we denote by \mathbb{X} the space of \mathbb{R}^2 -valued functions such that each component belongs to X . We introduce the spaces

$$\begin{aligned} \mathcal{C} &= \left\{ u \in [\mathcal{C}_c^\infty(\mathcal{O})]^2 \text{ such that } \operatorname{div} u = 0 \right\}, \\ \mathbb{V} &= \text{closure of } \mathcal{C} \text{ in } \mathbb{H}^1(\mathcal{O}), \end{aligned} \quad (2.4)$$

$$\mathbb{H} = \text{closure of } \mathcal{C} \text{ in } \mathbb{L}^2(\mathcal{O}).$$

We denote by (\cdot, \cdot) and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbb{L}^2(\mathcal{O})$ on \mathbb{H} , respectively. The inner product and the norm of $\mathbb{H}_0^1(\mathcal{O})$ are denoted respectively by $((\cdot, \cdot))$ and $\|\cdot\|$. We endow the space \mathbb{V} with the norm generated by the following scalar product

$$(u, v)_{\mathbb{V}} = (u, v) + \alpha((u, v)), \text{ for any } v \in \mathbb{V};$$

which is equivalent to $\|\cdot\|$, more precisely, we have

$$(\mathcal{P}^2 + \alpha)^{-1} \|v\|_{\mathbb{V}}^2 \leq \|v\|^2 \leq \alpha^{-1} \|v\|_{\mathbb{V}}^2, \text{ for any } v \in \mathbb{V}, \quad (2.5)$$

where \mathcal{P} is the constant from Poincaré's inequality.

We also introduce the following space

$$\mathbb{W} = \{u \in \mathbb{V} \text{ such that } \text{curl}(u - \alpha \Delta u) \in L^2(\mathcal{O})\},$$

and endow it with the norm generated by the scalar product

$$(u, v)_{\mathbb{W}} = (u, v)_{\mathbb{V}} + (\text{curl}(u - \alpha \Delta u), \text{curl}(v - \alpha \Delta v)). \quad (2.6)$$

The following result states that the norm induced by $(\cdot, \cdot)_{\mathbb{W}}$ is equivalent to the usual $\mathbb{H}^3(\mathcal{O})$ -norm on \mathbb{W} . This result can be found in [5], [4] and Lemma 2.1 in [16].

Lemma 2.1 *Set $\widetilde{\mathbb{W}} = \{v \in \mathbb{H}^3(\mathcal{O}) \text{ such that } \text{div} v = 0 \text{ and } v|_{\partial\mathcal{O}} = 0\}$. Then the following (algebraic and topological) identity holds:*

$$\mathbb{W} = \widetilde{\mathbb{W}}. \quad (2.7)$$

Moreover, there is a positive constant C such that

$$\|v\|_{\mathbb{H}^3(\mathcal{O})}^2 \leq C \left(\|v\|_{\mathbb{V}}^2 + |\text{curl}(v - \alpha \Delta v)|^2 \right), \quad (2.8)$$

for any $v \in \widetilde{\mathbb{W}}$.

From now on, we identify the space \mathbb{V} with its dual space \mathbb{V}^* via the Riesz representation, and we have the Gelfand triple

$$\mathbb{W} \subset \mathbb{V} \subset \mathbb{W}^*. \quad (2.9)$$

We denote by $\langle f, v \rangle$ the action of any element f of \mathbb{W}^* on an element $v \in \mathbb{W}$. It is easy to see

$$(v, w)_{\mathbb{V}} = \langle v, w \rangle, \quad \forall v \in \mathbb{V}, \quad \forall w \in \mathbb{W}.$$

Note that the injection of \mathbb{W} into \mathbb{V} is compact. Thus, there exists a sequence $\{e_i : i = 1, 2, 3, \dots\}$ of elements of \mathbb{W} which forms an orthonormal basis in \mathbb{W} . The elements of this sequence are the solutions of the eigenvalue problem

$$(v, e_i)_{\mathbb{W}} = \lambda_i (v, e_i)_{\mathbb{V}}, \text{ for any } v \in \mathbb{W}. \quad (2.10)$$

Here $\{\lambda_i : i = 1, 2, 3, \dots\}$ is an increasing sequence of positive eigenvalues. We have the following important result from [4] about the regularity of the functions e_i , $i = 1, 2, 3, \dots$.

Lemma 2.2 *Let \mathcal{O} be a bounded, simply-connected open subset of \mathbb{R}^2 with a boundary of class \mathcal{C}^3 , then the eigenfunctions of (2.10) belong to $\mathbb{H}^4(\mathcal{O})$.*

Consider the following “generalized Stokes equations”:

$$\begin{aligned} v - \alpha \Delta v &= f \text{ in } \mathcal{O}, \\ \operatorname{div} v &= 0 \text{ in } \mathcal{O}, \\ v &= 0 \text{ on } \partial\mathcal{O}. \end{aligned} \tag{2.11}$$

The following result can be found in [21], [22], Theorem 2.5 of [18] and Theorem 2.2 of [16].

Lemma 2.3 *Let \mathcal{O} be a connected, bounded open subset of \mathbb{R}^2 with boundary $\partial\mathcal{O}$ of class \mathcal{C}^l and let f be a function in \mathbb{H}^l , $l \geq 1$. Then the system (2.11) admits a solution $v \in \mathbb{H}^{l+2} \cap \mathbb{V}$. Moreover if f is an element of \mathbb{H} , then v is unique and the following relations hold*

$$(v, g)_{\mathbb{V}} = (f, g), \text{ for any } g \in \mathbb{V}, \tag{2.12}$$

and

$$\|v\|_{\mathbb{W}} \leq K \|f\|_{\mathbb{V}}. \tag{2.13}$$

Define the Stokes operator by

$$Au = -\mathbb{P}\Delta u, \quad \forall u \in D(A) = \mathbb{H}^2(\mathcal{O}) \cap \mathbb{V}, \tag{2.14}$$

here we denote by $\mathbb{P} : \mathbb{L}^2(\mathcal{O}) \rightarrow \mathbb{H}$ the usual Helmholtz-Leray projector. It follows from Lemma 2.3 that the operator $(I + \alpha A)^{-1}$ defines an isomorphism from $\mathbb{H}^l(\mathcal{O}) \cap \mathbb{H}$ into $\mathbb{H}^{l+2}(\mathcal{O}) \cap \mathbb{V}$ provided that \mathcal{O} is of class \mathcal{C}^l , $l \geq 1$. Moreover, the following properties hold

$$\begin{aligned} ((I + \alpha A)^{-1} f, g)_{\mathbb{V}} &= (f, g), \\ \|(I + \alpha A)^{-1} f\|_{\mathbb{W}} &\leq K \|f\|_{\mathbb{V}}, \end{aligned}$$

for any $f \in \mathbb{H}^l(\mathcal{O}) \cap \mathbb{V}$ and any $g \in \mathbb{V}$. From these facts, $\hat{A} = (I + \alpha A)^{-1} A$ defines a continuous linear operator from $\mathbb{H}^l(\mathcal{O}) \cap \mathbb{V}$ onto itself for $l \geq 2$, and satisfies

$$(\hat{A}u, g)_{\mathbb{V}} = (Au, g) = ((u, g)),$$

for any $u \in \mathbb{W}$ and $g \in \mathbb{V}$. Hence, for any $u \in \mathbb{W}$

$$(\hat{A}u, u)_{\mathbb{V}} = \|u\|.$$

Let

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

for any $u, v, w \in \mathcal{C}$. Then the following identity holds (see for instance [6] [1]):

$$((\operatorname{curl} \Phi) \times v, w) = b(v, \Phi, w) - b(w, \Phi, v), \tag{2.15}$$

for any smooth function Φ , v and w . Now we recall the following two lemmas which can be found in [16] (Lemma 2.3 and Lemma 2.4), and also in [6] [1].

Lemma 2.4 For any $u, v, w \in \mathbb{W}$, we have

$$|(\operatorname{curl}(u - \alpha \Delta u) \times v, w)| \leq \tilde{K} \|u\|_{\mathbb{H}^3} \|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}, \quad (2.16)$$

and

$$|(\operatorname{curl}(u - \alpha \Delta u) \times u, w)| \leq \Theta \|u\|_{\mathbb{V}}^2 \|w\|_{\mathbb{W}}. \quad (2.17)$$

Set

$$B(u, v) = \operatorname{curl}(u - \alpha \Delta u) \times v, \quad \forall u, v \in \mathbb{W}.$$

3 Formulation of the main result

In this section, we will state the precise assumptions on the coefficients and collect some preliminary results from [18] and [16], which will be used in the following sections.

Assume that $\{W(s), s \in [0, \infty)\}$ is a U -cylindrical Wiener process admitting the following representation:

$$W = \sum_n \beta_n \bar{e}_n,$$

where $(\bar{e}_n)_n$ is a complete orthonormal system of U and $(\beta_n)_n$ is a sequence of independent Brownian motions.

Given Hilbert spaces Q_1, Q_2 , we denote by $\mathcal{L}_2(Q_1, Q_2)$ the space of all Hilbert-Schmidt operators from Q_1 into Q_2 . And $\mathcal{L}(Q_1, Q_2)$ denotes the space of bounded linear operators from Q_1 into Q_2 . Let $\phi : \mathbb{V} \rightarrow \mathcal{L}_2(U, \mathbb{V})$ be a given measurable mapping. We denote by P_N the orthogonal projection from \mathbb{V} into the space $\operatorname{Span}(e_1, \dots, e_N)$. Now we introduce the following conditions:

(H0) The mapping $\phi : \mathbb{V} \rightarrow \mathcal{L}_2(U, \mathbb{V})$ is bounded and Lipschitz, i.e., there exist constants R and L_ϕ such that

$$R = \sup_{v \in \mathbb{V}} \|\phi(v)\|_{\mathcal{L}_2(U, \mathbb{V})}^2,$$

and

$$\|\phi(v_1) - \phi(v_2)\|_{\mathcal{L}_2(U, \mathbb{V})} \leq L_\phi \|v_1 - v_2\|_{\mathbb{V}}, \quad \forall v_1, v_2 \in \mathbb{V}.$$

(H1) Recall the constant K in Lemma 2.3, and Θ in (2.17). There exists $N \in \mathbb{N}$ and a bounded measurable mapping $g : \mathbb{V} \rightarrow \mathcal{L}(\mathbb{V}, U)$ such that for any $v \in \mathbb{V}$

$$\phi(v)g(v) = P_N, \quad (3.18)$$

and the viscosity constant ν satisfies

$$\frac{1}{2\Theta^2} \frac{\nu}{\mathcal{P}^2 + \alpha} \left(\frac{2\nu}{\mathcal{P}^2 + \alpha} - 1 - \frac{K^2 L_\phi^2}{\lambda_1} \right) \geq \left(1 + \frac{2}{\lambda_1} + \frac{2(\mathcal{P}^2 + \alpha)}{\lambda_1 \alpha^2} \right) K^2 R. \quad (3.19)$$

Remark 1 (3.18) can be seen as a non degeneracy condition on the low modes, and (3.19) is a technique condition.

Now we recall the concept of solution of the problem (1.1) in [16].

Definition 3.1 A stochastic process u is called a solution of the system (1.1), if

1. $u(0) = u_0$,
2. $u \in L^p(\Omega, \mathcal{F}, P; L^\infty([0, \infty), \mathbb{W}))$, $2 \leq p < \infty$,
3. For all $t \geq 0$, $u(t)$ is \mathcal{F}_t -measurable,
4. For any $t \in (0, \infty)$ and $v \in \mathbb{W}$, the following identity holds almost surely

$$\begin{aligned} & (u(t) - u(0), v)_{\mathbb{V}} + \int_0^t [\nu((u(s), v)) + (\text{curl}(u(s) - \alpha \Delta u(s)) \times u(s), v)] ds \\ &= \int_0^t (\phi(u(s)) dW(s), v). \end{aligned}$$

Using Galerkin approximation scheme for the system (1.1), Razafimandimby and Sango [16] obtained the following theorem (see Theorem 3.4 and Theorem 4.1 in [16]).

Theorem 3.2 Let $u_0 \in \mathbb{W}$. Assume (H0) holds. Then

- (1) the system (1.1) has a unique solution,
- (2) the solution u admits a version which is continuous in \mathbb{V} with respect to the strong topology and continuous in \mathbb{W} with respect to the weak topology.

Moreover, from the proof of Theorem 4.1 in [16], we have

Theorem 3.3 Assume that $u_1, u_2 \in \mathbb{W}$ are \mathcal{F}_t -measurable, and let $\{X_1(t+s), s \geq 0\}$ and $\{X_2(t+s), s \geq 0\}$ be two solutions of the system (1.1) with initial condition $X_1(t) = u_1$ and $X_2(t) = u_2$, respectively. Then, for any $O \in \mathcal{F}_t$

$$\begin{aligned} & E\left(\sigma(t+s, t) \|X_1(t+s) - X_2(t+s)\|_{\mathbb{V}}^2 1_O\right) \\ & \leq E\left(\|u_1 - u_2\|_{\mathbb{V}}^2 1_O\right) + C \int_t^{t+s} E\left(\sigma(l, t) \|X_1(l) - X_2(l)\|_{\mathbb{V}}^2 1_O\right) dl, \end{aligned} \quad (3.20)$$

here $\sigma(l, t) = \exp\left(-\int_t^l \|X_2(s)\|_{\mathbb{W}}^2 ds\right)$.

Remark 2 By (3.20), if $u_1 = u_2$ on $O \in \mathcal{F}_t$, then $X_1(t+\cdot) = X_2(t+\cdot)$ on O P -a.s..

For a \mathbb{W} -valued, \mathcal{F}_{t_0} -measurable random variable Y , let $u(t_0 + \cdot, t_0, Y)$ be the unique solution of (1.1) on the time interval $[0, \infty)$ with initial condition $u(t_0, t_0, Y) = Y$. Denote

$$X^x(t) = X(t, W, x) = \begin{cases} u(t, 0, x), & x \in \mathbb{W}; \\ x, & x \in \mathbb{V}/\mathbb{W}. \end{cases} \quad (3.21)$$

Then we define the operators $\mathcal{P}_t : B_b(\mathbb{V}) \rightarrow B_b(\mathbb{V})$ as

$$(\mathcal{P}_t \varphi)(x) = E[\varphi(X^x(t))],$$

where $B_b(\mathbb{V})$ is the space of bounded measurable functions on \mathbb{V} . Let $C_b(\mathbb{V})$ be the space of bounded continuous functions.

Lemma 3.1 $\{X^x, x \in \mathbb{V}\}$ defines a Markov process in the sense that, for every $x \in \mathbb{V}$, $\varphi \in C_b(\mathbb{V})$, $t, s > 0$

$$E[\varphi(X^x(t+s)) | \mathcal{F}_t] = (\mathcal{P}_s \varphi)(X^x(t)), \quad P\text{-a.s..} \quad (3.22)$$

Proof: Noticing that (3.22) holds when $x \in \mathbb{V}/\mathbb{W}$. Now given $x \in \mathbb{W}$, we have $X^x(t) = u(t, 0, x)$. To prove (3.22), it is sufficient to prove that

$$E[\varphi(u(t+s, 0, x))Z] = E[(\mathcal{P}_s\varphi)(u(t, 0, x))Z]$$

for every bounded \mathcal{F}_t -measurable r.v. Z .

Since, by Theorem 3.2,

$$u(t+s, 0, x) = u(s, t, u(t, 0, x)) \quad \text{and} \quad E(\|u(t, 0, x)\|_{\mathbb{W}}^2) < \infty,$$

it is sufficient to prove that

$$E[\varphi(u(t+s, t, \eta))Z] = E[(\mathcal{P}_s\varphi)(\eta)Z] \tag{3.23}$$

for every \mathbb{W} -valued \mathcal{F}_t -measurable r.v. η .

By (3.20), for any given $\xi_n, \xi \in \mathbb{W}$, the strong convergence of ξ_n to ξ in \mathbb{V} implies that $(\mathcal{P}_s\varphi)(\xi_n)$ converges to $(\mathcal{P}_s\varphi)(\xi)$. Hence, to prove (3.23), it is sufficient to prove it for every r.v. η of the form $\eta = \sum_{i=1}^k \eta^i 1_{A^i}$ with $\eta^i \in \mathbb{W}$ and $A^i \in \mathcal{F}_t$. By Remark 2, we just need to prove (3.23) for every deterministic $\eta \in \mathbb{W}$.

Now the r.v. $u(t+s, t, \eta)$ depends only on the increments of the Brownian motion between t and $t+s$, hence it is independent of \mathcal{F}_t . Therefore

$$E[\varphi(u(t+s, t, \eta))Z] = E[\varphi(u(t+s, t, \eta))]EZ.$$

Since $u(t+s, t, \eta)$ has the same law of $u(s, 0, \eta)$ (by uniqueness), we have $E[\varphi(u(t+s, t, \eta))] = E[\varphi(u(s, 0, \eta))]$ and thus

$$E[\varphi(u(t+s, t, \eta))Z] = E[\varphi(u(s, 0, \eta))]EZ = E[\varphi(u(s, 0, \eta))Z].$$

The proof is complete. ■

The space of probability measures on \mathbb{V} is denoted by $\mathcal{P}(\mathbb{V})$. The aim of this paper is to prove the following result.

Theorem 3.4 *Assume that (H0) holds, and (H1) holds with some $N \in \mathbb{N}$. Then there exists a unique invariant probability measure μ of $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$ on \mathbb{V} satisfying*

$$\int_{\mathbb{V}} \|u\|_{\mathbb{W}}^2 \mu(du) < \infty,$$

and there exist $C, \gamma' > 0$ such that for any $\lambda \in \mathcal{P}(\mathbb{V})$

$$\|\mathcal{P}_t^* \lambda - \mu\|_* \leq C e^{-\gamma' t} \left(1 + \int_{\mathbb{V}} \|u\|_{\mathbb{W}}^2 \lambda(du) \right).$$

4 Proof of the main result

This section is devoted to the proof of the main result. We first recall the general criterion established in [15].

Given a Polish space E , $Lip_b(E)$ will denote the space of all bounded, Lipschitz continuous functions on E . Set

$$\|\varphi\|_L = |\varphi|_{\infty} + L_{\varphi}, \quad \varphi \in Lip_b(E),$$

here $|\cdot|_\infty$ is the sup norm and L_φ is the Lipschitz constant of φ . The space of probability measures on E is denoted by $\mathcal{P}(E)$. It is endowed with the Wasserstein norm

$$\|\mu\|_* = \sup_{\varphi \in Lip_b(E), \|\varphi\|_L \leq 1} \left| \int_E \varphi(u) \mu(du) \right|, \quad \mu \in \mathcal{P}(E).$$

Let $(U, |\cdot|_U)$ and $(\mathbb{V}, \|\cdot\|_\mathbb{V})$ be the two Hilbert spaces introduced before. We consider a Markov process Υ living in \mathbb{V} and depending measurably on a cylindrical Wiener process W on U . Υ can be written as

$$\Upsilon(t) = \Upsilon(t, W, x_0),$$

where x_0 is the initial value $\Upsilon(0, W, x_0) = x_0$. We denote the distribution of $\Upsilon(\cdot, W, x_0)$ by $\mathcal{D}(\Upsilon(\cdot, W, x_0))$, and assume that $\mathcal{D}(\Upsilon(\cdot, W, x_0))$ is measurable with respect to x_0 . Let $(\mathcal{P}_t)_{t \geq 0}$ be the Markov transition semigroup associated with the Markov family $(\Upsilon(\cdot, W, x_0))_{x_0 \in \mathbb{V}}$.

The basis idea behind the criterion in [15] is to construct an auxiliary process $\tilde{\Upsilon}(t, W, x_0, \tilde{x}_0)$, which is “close” to $\Upsilon(t, W, x_0)$ and has a law absolutely continuous with respect to $\mathcal{D}(\Upsilon(\cdot, W, \tilde{x}_0))$. More precisely, suppose that there exists a function

$$\tilde{\Upsilon} : [0, \infty) \times C([0, \infty); \mathbb{R})^\mathbb{N} \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V},$$

satisfying the following conditions.

- (A) For every $x_0, \tilde{x}_0 \in \mathbb{V}$, $\tilde{\Upsilon}(\cdot, W, x_0, \tilde{x}_0)$ is non-anticipative and measurable with respect to W . Moreover,

$$(\Upsilon(t), \tilde{\Upsilon}(t)) = (\Upsilon(t, W, x_0), \tilde{\Upsilon}(t, W, x_0, \tilde{x}_0))$$

defines an homogenous Markov process and its law $\mathcal{D}(\Upsilon, \tilde{\Upsilon})$ is measurable with respect to (x_0, \tilde{x}_0) .

- (B) There exists a positive measurable function $\mathcal{H} : \mathbb{V} \rightarrow \mathbb{R}^+$ and a positive constant γ such that for any $x_0 \in \mathbb{V}$, $t \geq 0$, $\beta > 0$ and any stopping time $\tau \geq 0$, there exists $C_1, C'_\beta > 0$ satisfying

$$\begin{cases} E\left(\mathcal{H}(\Upsilon(t, W, x_0))\right) \leq e^{-\gamma t} \mathcal{H}(x_0) + C_1; \\ E\left(e^{-\beta \tau} \mathcal{H}(\Upsilon(\tau, W, x_0)) 1_{\tau < \infty}\right) \leq \mathcal{H}(x_0) + C'_\beta; \end{cases}$$

- (C) There exists a function $h : \mathbb{V} \times \mathbb{V} \rightarrow U$ such that for any $(t, x_0^1, x_0^2) \in [0, \infty) \times \mathbb{V} \times \mathbb{V}$ and cylindrical Wiener process W on U , we have almost surely

$$\tilde{\Upsilon}(t, W, x_0^1, x_0^2) = \Upsilon\left(t, W + \int_0^t h\left(\Upsilon(s, W, x_0^1), \tilde{\Upsilon}(s, W, x_0^1, x_0^2)\right) ds, x_0^2\right);$$

- (D) For any $x_0^1, x_0^2 \in \mathbb{V}$ satisfying

$$\mathcal{H}(x_0^1) + \mathcal{H}(x_0^2) \leq 2C_1, \tag{4.24}$$

and for any cylindrical Wiener processes W_1, W_2 on U , set

$$h(t) = h(\Upsilon(t, W_1, x_0^1), \tilde{\Upsilon}(t, W_1, x_0^1, x_0^2)),$$

there exists $\gamma_0 > 0$ such that

(D1) there exists $C > 0$ such that

$$\begin{aligned} & \mathbb{P}\left(|\Upsilon(t, W_2, x_0^2) - \Upsilon(t, W_1, x_0^1)|_{\mathbb{V}} \geq Ce^{-\gamma_0 t}, \tilde{\Upsilon}(\cdot, W_1, x_0^1, x_0^2) = \Upsilon(\cdot, W_2, x_0^2) \text{ on } [0, t]\right) \\ & \leq Ce^{-\gamma_0 t}, \quad \forall t \geq 0; \end{aligned}$$

(D2) for any $t_0 \geq 0$ and any stopping time $\tau \geq t_0$, we have

$$\begin{aligned} & \mathbb{P}\left(\int_{t_0}^{\tau} |h(t)|_U^2 dt \geq Ce^{-\gamma_0 t_0} \text{ and } \tilde{\Upsilon}(\cdot, W_1, x_0^1, x_0^2) = \Upsilon(\cdot, W_2, x_0^2) \text{ on } [0, \tau]\right) \\ & \leq Ce^{-\gamma_0 t_0}; \end{aligned}$$

(D3) there exists $p_1 > 0$ such that

$$\mathbb{P}\left(\int_0^{+\infty} |h(t)|_U^2 dt \leq C\right) \geq p_1.$$

Here is the criteria obtained [15].

Theorem 4.1 *Under the assumptions (A)–(D), there exists a unique stationary probability measure μ of $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$ on \mathbb{V} , satisfying*

$$\int_{\mathbb{V}} \mathcal{H}(u) d\mu(u) < \infty,$$

and there exist $C, \gamma' > 0$ such that for any $\lambda \in \mathcal{P}(\mathbb{V})$

$$\|\mathcal{P}_t^* \lambda - \mu\|_* \leq Ce^{-\gamma' t} \left(1 + \int_{\mathbb{V}} \mathcal{H}(u) d\lambda(u)\right).$$

4.1 The proof

As a part of the proof, we will prepare a number of estimates for the solutions of the equation (1.1).

From now on, we denote by C any generic constant which may change from one line to another.

Set $\mathbb{W}_M = \text{Span}(e_1, \dots, e_M)$. Let $u^M \in \mathbb{W}_M$ be the Galerkin approximations of (1.1) satisfying

$$\begin{aligned} & d(u^M, e_i)_{\mathbb{V}} + \nu((u^M, e_i))dt + b(u^M, u^M, e_i)dt - \alpha b(u^M, \Delta u^M, e_i)dt + \alpha b(e_i, \Delta u^M, u^M)dt \\ & = (\phi(u^M), e_i)dW(t), \quad i \in \{1, 2, \dots, M\}, \end{aligned} \tag{4.25}$$

where the notation $(\phi(u), e_i)$ stands for the operator in $\mathcal{L}(U, \mathbb{R})$ defined by

$$(\phi(u), e_i)h = (\phi(u)h, e_i), \quad \forall h \in U.$$

Then

$$\begin{aligned} \|(\phi(u), e_i)\|_{\mathcal{L}_2(U, \mathbb{R})}^2 &= \sum_{j=1}^{\infty} (\phi(u)\bar{e}_j, e_i)^2 \leq \|e_i\|_{\mathbb{H}}^2 \sum_{j=1}^{\infty} \|\phi(u)\bar{e}_j\|_{\mathbb{H}}^2 \\ &\leq C\|e_i\|_{\mathbb{H}}^2 \sum_{j=1}^{\infty} \|\phi(u)\bar{e}_j\|_{\mathbb{V}}^2 \leq C\|\phi(u)\|_{\mathcal{L}_2(U, \mathbb{V})}^2. \end{aligned}$$

We have the following result for u^M .

Lemma 4.1 Assume (H0) holds. There exist C_1 and C_β^1 only depending on ν , α , R and \mathcal{O} such that, for any $t \geq 0$, any $\beta > 0$ and any stopping time τ ,

$$E(\|u^M(t)\|_{\mathbb{W}}^2) \leq e^{-\frac{\nu t}{\mathcal{P}^2 + \alpha}} \|u_0^M\|_{\mathbb{W}}^2 + C_1 \quad (4.26)$$

and

$$E(e^{-\beta\tau} \|u^M(\tau)\|_{\mathbb{W}}^2 I_{\tau < \infty}) \leq \|u_0^M\|_{\mathbb{W}}^2 + C_\beta^1. \quad (4.27)$$

Proof: Applying Itô's formula, we have

$$\begin{aligned} & d(u^M, e_i)_{\mathbb{V}}^2 \\ & + 2(u^M, e_i)_{\mathbb{V}} \left[\nu((u^M, e_i)) + b(u^M, u^M, e_i) - \alpha b(u^M, \Delta u^M, e_i) + \alpha b(e_i, \Delta u^M, u^M) \right] dt \\ = & 2(u^M, e_i)_{\mathbb{V}} (\phi(u^M), e_i) dW(t) + |(\phi(u^M), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt. \end{aligned} \quad (4.28)$$

Notice that $\|u^M\|_{\mathbb{V}}^2 = \sum_{i=1}^M \lambda_i (u^M, e_i)_{\mathbb{V}}^2$. Multiplying by λ_i and taking summation over i , we get

$$d\|u^M\|_{\mathbb{V}}^2 + 2\nu\|u^M\|^2 dt = 2(\phi(u^M), u^M) dW(t) + \sum_{i=1}^M \lambda_i |(\phi(u^M), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt \quad (4.29)$$

here we used the fact that $b(u^M, u^M, u^M) = 0$.

Let $\tilde{G}(u^M(t))$ be the operator in $\mathcal{L}_2(U, \mathbb{W})$ defined as follows. For any $h \in U$, $\tilde{G}(u^M(t)) \times h \in \mathbb{W}$ is the unique solution of the following equation.

$$\begin{aligned} \tilde{G}(u^M(t)) \times h - \alpha \Delta (\tilde{G}(u^M(t)) \times h) &= \phi(u^M(t)) \times h \text{ in } \mathcal{O}, \\ \tilde{G}(u^M(t)) \times h &= 0 \text{ on } \partial\mathcal{O}. \end{aligned}$$

By Lemma 2.3, such a solution uniquely exists. Moreover,

$$(\tilde{G}(u^M(t)) \times h, e_i)_{\mathbb{V}} = (\phi(u^M(t)) \times h, e_i), \quad \forall i \in \{1, 2, \dots, M\}, \quad (4.30)$$

and there exists a positive constant K such that

$$\|\tilde{G}(u^M(t)) \times h\|_{\mathbb{W}} \leq K \|\phi(u^M(t)) \times h\|_{\mathbb{V}},$$

which implies that

$$\|\tilde{G}(u^M(t))\|_{\mathcal{L}_2(U, \mathbb{W})}^2 \leq K^2 \|\phi(u^M(t))\|_{\mathcal{L}_2(U, \mathbb{V})}^2.$$

Hence by (2.10),

$$\begin{aligned} & \sum_{i=1}^M \lambda_i |(\phi(u^M(t)), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 \\ = & \sum_{i=1}^M \lambda_i |(\tilde{G}(u^M(t)), e_i)_{\mathbb{V}}|_{\mathcal{L}_2(U, \mathbb{R})}^2 \\ = & \sum_{i=1}^M \frac{1}{\lambda_i} |(\tilde{G}(u^M(t)), e_i)_{\mathbb{W}}|_{\mathcal{L}_2(U, \mathbb{R})}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\lambda_1} \|\tilde{G}(u^M(t))\|_{\mathcal{L}_2(U, \mathbb{W})}^2 \\
&\leq \frac{K^2}{\lambda_1} \|\phi(u^M(t))\|_{\mathcal{L}_2(U, \mathbb{V})}^2 \\
&\leq \frac{K^2}{\lambda_1} R.
\end{aligned} \tag{4.31}$$

Combining this with (4.29), we obtain

$$d\|u^M\|_{\mathbb{V}}^2 + 2\nu\|u^M\|^2 dt \leq 2(\phi(u^M), u^M)dW(t) + \frac{K^2}{\lambda_1} Rdt. \tag{4.32}$$

Denote $\|v\|_* = |\text{curl}(v - \alpha\Delta v)|$ for any $v \in \mathbb{W}$. Next we estimate $\|u^M\|_*$. Setting

$$\Psi(u^M) = -\nu\Delta u^M + \text{curl}(u^M - \alpha\Delta u^M) \times u^M,$$

we have

$$d(u^M, e_i)_{\mathbb{V}} + (\Psi(u^M), e_i)dt = (\phi(u^M), e_i)dW(t).$$

Noting $\Psi(u^M) \in \mathbb{H}^1(\mathcal{O})$. By Lemma 2.3, there exists unique solution $v^M \in \mathbb{W}$ satisfying

$$\begin{cases} v^M - \alpha\Delta v^M = \Psi(u^M) \text{ in } \mathcal{O}; \\ v^M = 0 \text{ on } \partial\mathcal{O}. \end{cases}$$

Moreover,

$$(v^M, e_i)_{\mathbb{V}} = (\Psi(u^M), e_i), \quad \forall i \in \{1, 2, \dots, M\}.$$

Thus

$$d(u^M, e_i)_{\mathbb{V}} + (v^M, e_i)_{\mathbb{V}}dt = (\phi(u^M), e_i)dW(t). \tag{4.33}$$

By (4.30) and (2.10), we have

$$\lambda_i(\phi(u^M), e_i) = (\tilde{G}(u^M), e_i)_{\mathbb{W}}.$$

Multiplying λ_i to (4.33), it follows that

$$d(u^M, e_i)_{\mathbb{W}} + (v^M, e_i)_{\mathbb{W}}dt = (\tilde{G}(u^M), e_i)_{\mathbb{W}}dW(t).$$

Applying Itô's formula, we have

$$\begin{aligned}
&d(u^M, e_i)_{\mathbb{W}}^2 + 2(u^M, e_i)_{\mathbb{W}}(v^M, e_i)_{\mathbb{W}}dt \\
&= 2(u^M, e_i)_{\mathbb{W}}(\tilde{G}(u^M), e_i)_{\mathbb{W}}dW(t) + |(\tilde{G}(u^M), e_i)_{\mathbb{W}}|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt,
\end{aligned}$$

and hence

$$\begin{aligned}
&d\|u^M\|_{\mathbb{W}}^2 + 2(v^M, u^M)_{\mathbb{W}}dt \\
&= 2(\tilde{G}(u^M), u^M)_{\mathbb{W}}dW(t) + \sum_{i=1}^M |(\tilde{G}(u^M), e_i)_{\mathbb{W}}|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt.
\end{aligned}$$

By (2.6) we rewrite the above equation as follows

$$d[\|u^M\|_{\mathbb{V}}^2 + \|u^M\|_*^2] + 2\left[(v^M, u^M)_{\mathbb{V}} + \left(\text{curl}(u^M - \alpha\Delta u^M), \text{curl}(v^M - \alpha\Delta v^M)\right)\right]dt$$

$$\begin{aligned}
&= 2(\tilde{G}(u^M), u^M)_{\mathbb{V}} dW(t) + \sum_{i=1}^M |(\tilde{G}(u^M), e_i)_{\mathbb{W}}|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt \\
&\quad + 2\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\tilde{G}(u^M) - \alpha \Delta \tilde{G}(u^M))\right) dW(t).
\end{aligned}$$

Using the definition of v^M and \tilde{G} (see (4.30)), we obtain

$$\begin{aligned}
&d[\|u^M\|_{\mathbb{V}}^2 + \|u^M\|_*^2] + 2\left[(\Psi(u^M), u^M) + \left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\Psi(u^M))\right)\right] dt \\
&= 2(\phi(u^M), u^M) dW(t) + \sum_{i=1}^M \lambda_i^2 |(\phi(u^M), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt \\
&\quad + 2\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\phi(u^M))\right) dW(t).
\end{aligned}$$

Subtracting (4.29) from the above equation, we obtain

$$\begin{aligned}
&d\|u^M\|_*^2 + 2\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\Psi(u^M))\right) dt \\
&= \sum_{i=1}^M (\lambda_i^2 - \lambda_i) |(\phi(u^M), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt \\
&\quad + 2\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\phi(u^M))\right) dW(t),
\end{aligned} \tag{4.34}$$

here we used the fact $2(\Psi(u^M), u^M) = 2\nu\|u^M\|^2$.

Since

$$\operatorname{curl}\left(\operatorname{curl}(u^M - \alpha \Delta u^M) \times u^M\right) = (u^M \cdot \nabla)(\operatorname{curl}(u^M - \alpha \Delta u^M)),$$

we have $\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}\left(\operatorname{curl}(u^M - \alpha \Delta u^M) \times u^M\right)\right) = 0$. Hence

$$\begin{aligned}
&\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\Psi(u^M))\right) \\
&= \left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(-\nu \Delta u^M)\right) \\
&= \frac{\nu}{\alpha} \|u^M\|_*^2 - \frac{\nu}{\alpha} \left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl} u^M\right).
\end{aligned}$$

It follows from (4.34) that

$$\begin{aligned}
&d\|u^M\|_*^2 + \frac{2\nu}{\alpha} \|u^M\|_*^2 dt - \frac{2\nu}{\alpha} \left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl} u^M\right) dt \\
&= \sum_{i=1}^M (\lambda_i^2 - \lambda_i) |(\phi(u^M), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt \\
&\quad + 2\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl}(\phi(u^M))\right) dW(t).
\end{aligned} \tag{4.35}$$

Using the fact that

$$|\operatorname{curl}(u)|^2 \leq \frac{2}{\alpha} \|u\|_{\mathbb{V}}^2 \quad \text{for any } u \in \mathbb{W}, \tag{4.36}$$

we have

$$\left|\left(\operatorname{curl}(u^M - \alpha \Delta u^M), \operatorname{curl} u^M\right)\right|$$

$$\begin{aligned}
&\leq \sqrt{\frac{2}{\alpha}} \|u^M(s)\|_{\mathbb{V}} \|u^M(s)\|_* \\
&\leq \frac{1}{2} \|u^M(s)\|_*^2 ds + \frac{2}{\alpha} \|u^M(s)\|_{\mathbb{V}}^2.
\end{aligned} \tag{4.37}$$

Using similar arguments as that for (4.31), we have

$$\sum_{i=1}^M (\lambda_i^2 + \lambda_i) |(\phi(u^M(s)), e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 \leq (1 + \frac{1}{\lambda_1}) K^2 R. \tag{4.38}$$

Combining (4.35), (4.37) and (4.38), we arrive at

$$\begin{aligned}
d\|u^M\|_*^2 + \frac{\nu}{\alpha} \|u^M\|_*^2 dt &\leq \frac{4\nu}{\alpha^2} \|u^M\|_{\mathbb{V}}^2 dt + (1 + \frac{1}{\lambda_1}) K^2 R dt \\
&\quad + 2 \left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\phi(u^M)) \right) dW(t).
\end{aligned} \tag{4.39}$$

By (2.5), (4.32) and (4.39), we obtain

$$\begin{aligned}
d\|u^M\|_{\mathbb{W}}^2 + l \|u^M\|_{\mathbb{W}}^2 dt &\leq l_0 R dt + (2 + \frac{4(\mathcal{P}^2 + \alpha)}{\alpha^2}) (\phi(u^M), u^M) dW(t) \\
&\quad + 2 \left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\phi(u^M)) \right) dW(t),
\end{aligned} \tag{4.40}$$

here $l = \frac{\nu}{\mathcal{P}^2 + \alpha}$, $l_0 = (1 + \frac{2}{\lambda_1} + \frac{2(\mathcal{P}^2 + \alpha)}{\lambda_1 \alpha^2}) K^2$.

Applying chain rule to $e^{lt} \|u^M\|_{\mathbb{W}}^2$ and taking the expectation, we obtain

$$E \|u^M(t)\|_{\mathbb{W}}^2 \leq e^{-lt} \|u^M(0)\|_{\mathbb{W}}^2 + \frac{l_0 R}{l},$$

which is the desired inequality (4.26).

Let $\beta > 0$ and τ be a stopping time. Applying Itô's formula to $e^{-\beta t} \|u^M(t)\|_{\mathbb{W}}^2$, we have

$$\begin{aligned}
&d(e^{-\beta t} \|u^M\|_{\mathbb{W}}^2) + e^{-\beta t} (\beta + l) \|u^M\|_{\mathbb{W}}^2 dt \\
&\leq e^{-\beta t} l_0 R dt + (2 + \frac{4(\mathcal{P}^2 + \alpha)}{\alpha^2}) e^{-\beta t} (\phi(u^M), u^M) dW(t) \\
&\quad + 2e^{-\beta t} \left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\phi(u^M)) \right) dW(t).
\end{aligned} \tag{4.41}$$

This implies that, for any $n \in \mathbb{N}$,

$$E \left(e^{-\beta(\tau \wedge n)} \|u^M(\tau \wedge n)\|_{\mathbb{W}}^2 \right) \leq \|u^M(0)\|_{\mathbb{W}}^2 + \frac{l_0 R}{\beta}. \tag{4.42}$$

Letting $n \rightarrow \infty$, we obtain (4.27).

The proof is complete. ■

Denote by $u(\cdot, W, u_0)$ the unique solution of (1.1) with initial value u_0 . By similar arguments as in the proof of Theorem 4.2 in [16], we have the following lemma which will be used later.

Lemma 4.2 *The sequence of Galerkin approximations $(u^M)_{M \geq 1}$ satisfies*

$$\lim_{M \rightarrow \infty} E(\|u^M(t) - u(t)\|_{\mathbb{V}}^2) = 0, \quad t > 0,$$

$$\lim_{M \rightarrow \infty} E\left(\int_0^T \|u^M(t) - u(t)\|_{\mathbb{V}}^2 dt\right) = 0.$$

By Lemma 4.1, we deduce that

$$u^M(t) \rightarrow u(t, W, u_0) \text{ weakly in } L^2(\Omega, \mathcal{F}, P; \mathbb{W}).$$

Furthermore, the following result holds.

Proposition 4.2 *Assume (H0) holds. There exists constant C_1 only depending on ν , α , R and \mathcal{O} such that , for any $t \geq 0$,*

$$E(\|u(t, W, u_0)\|_{\mathbb{W}}^2) \leq e^{-\frac{\nu t}{\mathcal{P}^2 + \alpha}} \|u_0\|_{\mathbb{W}}^2 + C_1. \quad (4.43)$$

Moreover, for any $\beta > 0$, there exists a constant C_β^1 such that for any stopping time τ ,

$$E(e^{-\beta\tau} \|u(\tau, W, u_0)\|_{\mathbb{W}}^2 I_{\tau < \infty}) \leq \|u_0\|_{\mathbb{W}}^2 + C_\beta^1. \quad (4.44)$$

Recall $l = \frac{\nu}{\mathcal{P}^2 + \alpha}$ and $l_0 = (1 + \frac{2}{\lambda_1} + \frac{2(\mathcal{P}^2 + \alpha)}{\lambda_1 \alpha^2})K^2$. Define the energy functional

$$E_f(t) = \|f(t)\|_{\mathbb{W}}^2 + \frac{l}{2} \int_0^t \|f(s)\|_{\mathbb{W}}^2 ds.$$

Lemma 4.3 *Assume (H0) holds. There exists $\kappa > 0$ such that for any $\kappa_0 \leq \kappa/2$*

$$E\left[\exp\left(\kappa_0 \sup_{t \geq 0} (E_{u^M(\cdot, W, u_0^M)}(t) - l_0 R t)\right)\right] \leq 2 \exp(\kappa_0 \|u_0^M(0)\|_{\mathbb{W}}^2). \quad (4.45)$$

Proof: Set

$$\mathcal{A}(t) = \int_0^t \left(2 + \frac{4(\mathcal{P}^2 + \alpha)}{\alpha^2}\right) (\phi(u^M), u^M) dW(s) + 2 \int_0^t \left(\text{curl}(u^M - \alpha \Delta u^M), \text{curl}(\phi(u^M))\right) dW(s).$$

It is easy to see that

$$d\langle \mathcal{A} \rangle(t) \leq cR \|u^M\|_{\mathbb{W}}^2 dt.$$

Let $\kappa = \frac{l}{cR}$ and

$$\mathcal{A}_\kappa(t) = \mathcal{A}(t) - \frac{\kappa}{2} \langle \mathcal{A} \rangle(t).$$

By (4.40), we have

$$E_{u^M}(t) \leq \|u^M(0)\|_{\mathbb{W}}^2 + l_0 R t + \mathcal{A}_\kappa(t). \quad (4.46)$$

Since $e^{\kappa \mathcal{A}_\kappa}$ is a positive supermartingale whose value is 1 at $t = 0$, we have

$$P\left(\sup_{t \geq 0} \mathcal{A}_\kappa(t) \geq \rho\right) \leq P\left(\sup_{t \geq 0} \exp(\kappa \mathcal{A}_\kappa(t)) \geq \exp(\kappa \rho)\right) \leq \exp(-\kappa \rho), \quad (4.47)$$

which implies, letting $\kappa_0 \leq \kappa/2$,

$$E(e^{\kappa_0 \sup \mathcal{A}_\kappa}) = 1 + \kappa_0 \int_0^\infty e^{\kappa_0 y} P\left(\sup_{t \geq 0} \mathcal{A}_\kappa(t) \geq y\right) dy \leq 2. \quad (4.48)$$

Combining (4.48) and (4.46), we deduce (4.45). ■

Setting $\rho > 0$. Let $\tilde{u}^M \in \mathbb{W}_M$ be the solution of the following SPDE

$$\begin{aligned} & d(\tilde{u}^M, e_i)_\mathbb{V} + \nu((\tilde{u}^M, e_i))dt + b(\tilde{u}^M, \tilde{u}^M, e_i)dt - \alpha b(\tilde{u}^M, \Delta \tilde{u}^M, e_i)dt + \alpha b(e_i, \Delta \tilde{u}^M, \tilde{u}^M)dt \\ = & (\rho P_N(\tilde{u}^M - u^M(t, W, u_0)), e_i)_\mathbb{V}dt + (\phi(\tilde{u}^M), e_i)dW(t) \end{aligned} \quad (4.49)$$

for any $i \in \{1, 2, \dots, M\}$, with initial value $\tilde{u}^M(0) = P_M \tilde{u}_0$, $\tilde{u}_0 \in W$.

Lemma 4.4 *Assume that (H0) and (H1) hold. There exist $\varpi > 0$ and $\kappa_0 \geq 0$ such that*

$$\begin{aligned} & \sup_{t \geq 0} E\left(\left(e^t \|r_M(t)\|_\mathbb{V}^2 + \int_0^t e^s \|r_M(s)\|_\mathbb{V}^2 ds\right)^\varpi\right) \\ & \leq 2 \|r_M(0)\|_\mathbb{V}^{2\varpi} \exp(\kappa_0 \|u^M(0)\|_\mathbb{W}^2) \end{aligned} \quad (4.50)$$

where $r_M = \tilde{u}^M - u^M$.

Proof: Note that

$$d(r_M, e_i)_\mathbb{V} + \nu((r_M, e_i))dt + (\delta B, e_i)dt + \rho(P_N r_M, e_i)_\mathbb{V}dt = (\delta \phi, e_i)dW(s),$$

here $\delta B = B(\tilde{u}^M, \tilde{u}^M) - B(u^M, u^M)$ and $\delta \phi = \phi(\tilde{u}^M) - \phi(u^M)$.

Applying Itô's formula to $(r_M, e_i)_\mathbb{V}^2$, and remembering that $\|r_M\|_\mathbb{V}^2 = \sum_{i=1}^M \lambda_i (r_M, e_i)_\mathbb{V}^2$, we have

$$\begin{aligned} & d\|r_M\|_\mathbb{V}^2 + 2\nu\|r_M\|^2 dt + 2(\delta B, r_M)dt + 2(\rho P_N r_M, r_M)_\mathbb{V}dt \\ = & 2(\delta \phi, r_M)dW(t) + \sum_{i=1}^M \lambda_i |(\delta \phi, e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 dt. \end{aligned}$$

By (2.17),

$$|2(\delta B, r_M)| = |2(B(r_M, r_M) - B(r_M, u^M))| \leq 2\Theta \|r_M\|_\mathbb{V}^2 \|u^M\|_\mathbb{W} \leq \|r_M\|_\mathbb{V}^2 + \Theta^2 \|r_M\|_\mathbb{V}^2 \|u^M\|_\mathbb{W}^2.$$

By the similar arguments as that for the proof of (4.31),

$$\sum_{i=1}^M \lambda_i |(\delta \phi, e_i)|_{\mathcal{L}_2(U, \mathbb{R})}^2 \leq \frac{K^2}{\lambda_1} |\delta \phi|_{\mathcal{L}_2(U, \mathbb{V})}^2 \leq \frac{K^2 L_\phi^2}{\lambda_1} \|r_M\|_\mathbb{V}^2.$$

Set $l_1 = \frac{2\nu}{\mathcal{P}^2 + \alpha} - 1 - \frac{K^2 L_\phi^2}{\lambda_1} > 0$ and setting $\Lambda_1 = \Theta^2$, we have

$$d\|r_M\|_\mathbb{V}^2 + \left(l_1 - \Lambda_1 \|u^M\|_\mathbb{W}^2\right) \|r_M\|_\mathbb{V}^2 dt \leq 2(\delta \phi, r_M)dW(t).$$

Set $G_1(t) = e^{-l_1 t + \Lambda_1 \int_0^t \|u^M(s)\|_{\mathbb{W}}^2 ds}$. By the chain rule, we have

$$d(e^t G_1^{-1}(t) \|r_M(t)\|_{\mathbb{V}}^2) + e^t G_1^{-1}(t) \|r_M(t)\|_{\mathbb{V}}^2 dt \leq 2e^t G_1^{-1}(t) (\delta\phi, r_M) dW(t).$$

Integrating the above inequality and taking expectation, it follows that

$$E\left(e^t G_1^{-1}(t) \|r_M(t)\|_{\mathbb{V}}^2 + \int_0^t e^s G_1^{-1}(s) \|r_M(s)\|_{\mathbb{V}}^2 ds\right) \leq \|r_M(0)\|_{\mathbb{V}}^2. \quad (4.51)$$

By Hölder inequality,

$$\begin{aligned} & E\left(\left(e^t \|r_M(t)\|_{\mathbb{V}}^2 + \int_0^t e^s \|r_M(s)\|_{\mathbb{V}}^2 ds\right)^\varpi\right) \\ & \leq \sqrt{E(\sup_{t \geq 0} G_1^{2\varpi}(t))} \left(E\left(e^t G_1^{-1}(t) \|r_M(t)\|_{\mathbb{V}}^2 + \int_0^t e^s G_1^{-1}(s) \|r_M(s)\|_{\mathbb{V}}^2 ds\right)\right)^\varpi. \end{aligned} \quad (4.52)$$

Choosing $\varpi > 0$ sufficiently small, it follows from Lemma 4.3 and condition **(H1)** that

$$\begin{aligned} E(\sup_{t \geq 0} G_1^{2\varpi}(t)) &= E(\sup_{t \geq 0} e^{-2\varpi l_1 t + 2\Lambda_1 \varpi \int_0^t \|u^M(s)\|_{\mathbb{W}}^2 ds}) \\ &\leq E(\sup_{t \geq 0} e^{\frac{4\varpi \Lambda_1}{t} (E_{u^M}(t) - \frac{l_1 l}{2\Lambda_1} t)}) \\ &\leq E\left(\exp\left(\sup_{t \geq 0} \frac{4\varpi \Lambda_1}{l} (E_{u^M}(t) - \frac{l_1 l}{2\Lambda_1} t)\right)\right). \end{aligned} \quad (4.53)$$

(4.51) (4.52) and (4.53) imply (4.50). The proof is complete. \blacksquare

Recall X^x defined by (3.21). By Theorem 3.4 and Theorem 4.1 in [16], there exists a unique solution $\tilde{u} = \tilde{u}(\cdot, W, u_0, \tilde{u}_0)$ satisfying

$$\begin{aligned} & d(\tilde{u} - \alpha \Delta \tilde{u}) + \left(-\nu \Delta \tilde{u} + \operatorname{curl}(\tilde{u} - \alpha \Delta \tilde{u}) \times \tilde{u}\right) dt + \rho P_N((\tilde{u} - X^{u_0}) - \alpha \Delta(\tilde{u} - X^{u_0})) dt \\ & = \phi(\tilde{u}) dW, \end{aligned} \quad (4.54)$$

with initial value $\tilde{u}(0) = \tilde{u}_0 \in \mathbb{W}$.

By Lemma 4.2 and using similar arguments as Theorem 4.2 in [16], we have

Lemma 4.5 *The sequence of approximations $(\tilde{u}^M)_{M \geq 1}$ satisfies*

$$\begin{aligned} & \lim_{M \rightarrow \infty} E(\|\tilde{u}^M(t) - \tilde{u}(t)\|_{\mathbb{V}}^2) = 0, \\ & \lim_{M \rightarrow \infty} E\left(\int_0^T \|\tilde{u}^M(t) - \tilde{u}(t)\|_{\mathbb{V}}^2 dt\right) = 0. \end{aligned}$$

Combining Lemma 4.2, Lemma 4.5 and Lemma 4.4 and applying Fatou's lemma, we have

Proposition 4.3 *Assume that **(H0)** and **(H1)** hold. There exist $\varpi > 0$ and $\kappa_0 > 0$ such that*

$$\begin{aligned} & E\left(\left(e^t \|r(t)\|_{\mathbb{V}}^2 + \int_0^t e^s \|r(s)\|_{\mathbb{V}}^2 ds\right)^\varpi\right) \\ & \leq 2\|r(0)\|_{\mathbb{V}}^{2\varpi} \exp(\kappa_0 \|u_0\|_{\mathbb{W}}^2) \end{aligned} \quad (4.55)$$

where $r(t) = \tilde{u}(t) - u(t)$.

4.2 Completion of the proof

Now we verify the assumptions **(A)**-**(D)** of Theorem 4.1. Denote the solution of (4.54) by $\tilde{u}(t, W, u_0, \tilde{u}_0)$. Set

$$\tilde{X}(t, W, x_0, \tilde{x}_0) = \begin{cases} \tilde{u}(t, W, x_0, \tilde{x}_0), & \tilde{x}_0 \in \mathbb{W}; \\ \tilde{x}_0, & \tilde{x}_0 \in \mathbb{V}/\mathbb{W}, \end{cases} \quad (4.56)$$

and recall (3.21). We set $(\Upsilon(t), \tilde{\Upsilon}(t)) = (X(t, W, x_0), \tilde{X}(t, W, x_0, \tilde{x}_0))$. Then **(A)** is a consequence of the well-posedness of the equations.

We set $\mathcal{H} = \|\cdot\|_{\mathbb{W}}^2$. By Proposition 4.2, we have **(B)**. Let $h(u_0, u_1) = -g(u_1)\rho P_N(u_1 - u_0)$ and recall **(H1)**, then **(C)** holds.

By **(H1)**, we have

1.

$$\begin{aligned} & P\left(|\Upsilon(t, W_2, x_0^2) - \Upsilon(t, W_1, x_0^1)|_{\mathbb{V}} \geq Ce^{-\gamma_0 t}, \tilde{\Upsilon}(\cdot, W_1, x_0^1, x_0^2) = \Upsilon(\cdot, W_2, x_0^2) \text{ on } [0, t]\right) \\ &= P(\|r(t)\|_{\mathbb{V}} \geq Ce^{-\gamma_0 t}); \end{aligned}$$

2. for any $t_0 \geq 0$ and any stopping time $\tau \geq t_0$,

$$\begin{aligned} & P\left(\int_{t_0}^{\tau} |h(t)|_{\mathbb{U}}^2 dt \geq Ce^{-\gamma_0 t_0} \text{ and } \tilde{\Upsilon}(\cdot, W_1, x_0^1, x_0^2) = \Upsilon(\cdot, W_2, x_0^2) \text{ on } [0, \tau]\right) \\ & \leq P\left(\int_{t_0}^{\tau} \|r(t)\|_{\mathbb{V}}^2 dt \geq \tilde{C}e^{-\gamma_0 t_0}\right); \end{aligned}$$

3.

$$P\left(\int_0^{+\infty} |h(t)|_{\mathbb{U}}^2 dt \leq C\right) \geq P\left(\int_0^{+\infty} \|r(t)\|_{\mathbb{V}}^2 dt \leq \tilde{C}\right).$$

By Chebyshev inequality, we deduce **(D)** directly from Proposition 4.3.

Thus, we can apply Theorem 4.1 to obtain our main result Theorem 3.4.

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